MONOMIAL IDEALS OF MINIMAL DEPTH AND TRIVIAL MODIFICATIONS

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Abstract. Let S be a polynomial algebra over a field. We study classes of monomial ideals (as for example lexsegment ideals) of S having minimal depth. In particular, Stanley's conjecture holds for these ideals. Also we show that if Stanley's conjecture holds for a square free monomial ideal then it holds for all its trivial modifications.

Key Words: Monomial ideal, Stanley decomposition, Stanley depth, Lexsegment ideal, Minimal depth.
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Introduction

Let K be a field and $S = K[x_1, \ldots, x_n]$ be a polynomial ring in n variables over K. Let $I \subset S$ be a monomial ideal and $I = \cap_{i=1}^{s} Q_i$ an irredundant primary decomposition of I, where the Q_i are monomial ideals. Let Q_i be P_i-primary. Then each P_i is a monomial prime ideal and $\mathrm{Ass}(S/I) = \{P_1, \ldots, P_s\}$.

According to Lyubeznik [13] the size of I, denoted size(I), is the number $a+(n-b)-1$, where a is the minimum number t such that there exist $j_1 < \cdots < j_t$ with

$$\sqrt{\sum_{l=1}^{t} Q_{j_l}} = \sqrt{\sum_{j=1}^{s} Q_j},$$

and where $b = \mathrm{ht}(\sum_{j=1}^{s} Q_j)$. It is clear from the definition that size(I) depends only on the associated prime ideals of S/I. In the above definition if we replaced "there exists $j_1 < \cdots < j_t$" by "for all $j_1 < \cdots < j_t$", we obtain the definition of bigsize(I), introduced by Popescu [15]. Clearly bigsize$(I) \geq$ size(I).

Theorem 0.1. *(Lyubeznik [13]) Let $I \subset S$ ba a monomial ideal then* $\mathrm{depth}(I) \geq 1 + \mathrm{size}(I)$.

Herzog, Popescu and Vladoiu say in [7] that a monomial ideal I has *minimal depth*, if $\mathrm{depth}(I) = \mathrm{size}(I) + 1$. Suppose above that $P_i \not\subset \sum_{1=j\neq i}^{s} P_j$ for all $i \in [s]$. Then I has minimal depth as shows our Corollary 1.3 which extends [15, Theorem 2.3]. It is easy to see that if I has bigsize 1 then it must have minimal depth (see our Corollary 1.5).

Next we consider the lexicographical order on the monomials of S induced by $x_1 > x_2 > \cdots > x_n$. Let $d \geq 2$ be an integer and $\mathcal{M}_d$ the set of monomials of degree

d of S. For two monomials $u, v \in \mathcal{M}_d$, with $u \geq_{lex} v$, the set

$$\mathcal{L}(u,v) = \{w \in \mathcal{M}_d | u \geq_{lex} w \geq_{lex} v\}$$

is called a lexsegment set. A lexsegment ideal in S is a monomial ideal of S which is generated by a lexsegment set. We show that a lexsegment ideal has minimal depth (see our Theorem 1.6).

Now, let M be a finitely generated multigraded S-module. Let M be an S-module, $z \in M$ be a homogeneous element in M and $zK[Z]$, $Z \subseteq \{x_1, \ldots, x_n\}$ the linear K-subspace of M of all elements zf, $f \in K[Z]$. Such a linear K-subspace $zK[Z]$ is called a Stanley space of dimension $|Z|$ if it is a free $K[Z]$-module, where $|Z|$ denotes the number of indeterminates in Z. A presentation of M as a finite direct sum of spaces $\mathcal{D}: \ M = \bigoplus_{i=1}^{r} z_i K[Z_i]$ is called a Stanley decomposition. Stanley depth of a decomposition $\mathcal{D}$ is the number

$$\operatorname{sdepth} \mathcal{D} = \min\{|Z_i| : i = 1, \ldots, r\}.$$

The number

$$\operatorname{sdepth}(M) := \max\{\operatorname{sdepth}(\mathcal{D}) : \text{Stanley decomposition of } M\}$$

is called Stanley depth of M. In [17] R. P. Stanley conjectured that

$$\operatorname{sdepth}(M) \geq \operatorname{depth}(M).$$

Theorem 0.2 ([7])**.** *Let $I \subset S$ be a monomial ideal then* $\operatorname{sdepth}(I) \geq 1 + \operatorname{size}(I)$. *In particular, Stanley's conjecture holds for the monomial ideals of minimal depth.*

As a consequence, Stanley's depth holds for all ideals considered above since they have minimal depth. It is still not known a relation between $\operatorname{sdepth}(I)$ and $\operatorname{sdepth}(S/I)$, but our Theorem 2.3 shows that Stanley's conjecture holds also for S/I if $P_i \not\subset \sum_{1=j\neq i}^{s} P_j$ for all $i \in [s]$.

Let J be a so called trivial modification of a square free monomial ideal I in the sense of [8], [1]. Our Theorem 2.6 shows that $\operatorname{sdepth}(J) = \operatorname{sdepth}(I)$. It follows that if Stanley's conjecture holds for I then it holds for all trivial modifications of it (see our Corollary 2.7).

1. Minimal depth

We start this section extending some results of Popescu in [15]. Lemma 1.1, Proposition 1.2, Lemma 1.4 and Corollary 1.5 were proved by Popescu when I is a squarefree monomial ideal. We show that with some small changes the same proofs work even in the non-squarefree case.

Lemma 1.1. *Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant presentation of I as an intersection of primary monomial ideals. Let $P_i := \sqrt{Q_i}$. If $P_s \not\subset \sum_{i=1}^{s-1} P_i$, then*

$$\operatorname{depth}(S/I) = \min\{\operatorname{depth}(S/\cap_{i=1}^{s-1} Q_i), \operatorname{depth}(S/Q_s),\ 1 + \operatorname{depth}(S/\cap_{i=1}^{s-1} (Q_i + Q_s))\}.$$

Proof. We have the following exact sequence

$$0 \longrightarrow S/I \longrightarrow S/\cap_{i=1}^{s-1} Q_i \oplus S/Q_s \longrightarrow S/\cap_{i=1}^{s-1} (Q_i + Q_s) \longrightarrow 0.$$

Clearly $\operatorname{depth}(S/I) \le \operatorname{depth}(S/Q_s)$ by [2, Proposition 1.2.13]. Choosing x_j^a where $x_j \in P_s \not\subset \sum_{i=1}^{s-1} P_i$ and a is minimum such that $x_j^a \in Q_s$ we see that $I : x_j^a = \cap_{i=1}^{s-1} Q_i$ and by [16, Corollary 1.3] we have

$$\operatorname{depth}(S/I) \le \operatorname{depth} S/(I : x_j^a) = \operatorname{depth} S/(\cap_{i=1}^{s-1} Q_i).$$

Now by using Depth Lemma (see [18, Lemma 1.3.9]) we have

$$\operatorname{depth}(S/I) = \min\{\operatorname{depth}(S/\cap_{i=1}^{s-1} Q_i), \operatorname{depth}(S/Q_s),\ 1 + \operatorname{depth}(S/\cap_{i=1}^{s-1} (Q_i + Q_s))\},$$

which is enough. □

Proposition 1.2. *Let* $I = \bigcap_{i=1}^{s} Q_i$ *be the irredundant presentation of* I *as an intersection of primary monomial ideals. Let* $P_i := \sqrt{Q_i}$*. If* $P_i \not\subset \sum_{1=i\neq j}^{s-1} P_j$ *for all* $i \in [s]$*. Then* $\operatorname{depth}(S/I) = s - 1$*.*

Proof. It is enough to consider the case when $\sum_{j=1}^{s} P_j = \mathfrak{m}$. We use induction on s. If $s = 1$ the result is trivial. Suppose that $s > 1$. By Lemma 1.1 we get

$$\operatorname{depth}(S/I) = \min\{\operatorname{depth}(S/\cap_{i=1}^{s-1} Q_i), \operatorname{depth}(S/Q_s),\ 1 + \operatorname{depth}(S/\cap_{i=1}^{s-1} (Q_i + Q_s))\}.$$

Then by induction hypothesis we have

$$\operatorname{depth}(S/\cap_{i=1}^{s-1} Q_i) = s - 2 + \dim(S/(\sum_{i=1}^{s-1} Q_i)) \ge s - 1.$$

We see that $\cap_{i=1}^{s-1}(Q_i + Q_s)$ satisfies also our assumption, the induction hypothesis gives $\operatorname{depth}(S/\cap_{i=1}^{s-1} (Q_i + Q_s)) = s - 2$. Since $Q_i \not\subset Q_s$, $i < s$ by our assumption we get $\operatorname{depth}(S/Q_s) > \operatorname{depth}(S/(Q_i + Q_s))$ for all $i < s$. It follows $\operatorname{depth}(S/Q_s) \ge 1 + \operatorname{depth}(S/\cap_{i=1}^{s-1} (Q_i + Q_s))$ which is enough. □

Corollary 1.3. *Let* $I \subset S$ *be a monomial ideal such that* $\operatorname{Ass}(S/I) = \{P_1 \dots, P_s\}$ *where* $P_i \not\subset \sum_{1=j\neq i}^{s} P_j$ *for all* $i \in [s]$*. Then* I *has minimal depth.*

Proof. Clearly $\operatorname{size}(I) = s - 1$ and by Proposition 1.2 we have $\operatorname{depth}(I) = s$, thus we have $\operatorname{depth}(I) = \operatorname{size}(I) + 1$, i.e. I has minimal depth. □

Lemma 1.4. *Let* $I = \cap_{i=1}^{s} Q_i$ *be the irredundant primary decomposition of* I *and* $\sqrt{Q_i} \neq \mathfrak{m}$ *for all* i*. Suppose that there exists* $1 \le r < s$ *such that* $\sqrt{Q_i + Q_j} = \mathfrak{m}$ *for each* $r < j \le s$ *and* $1 \le i \le r$*. Then* $\operatorname{depth}(I) = 2$*.*

Proof. The proof follows by using Depth Lemma on the following exact sequence.

$$0 \longrightarrow S/I \longrightarrow S/\cap_{i=1}^{r} Q_i \oplus S/\cap_{j>r}^{s} Q_j \longrightarrow S/\cap_{i=1}^{r} \cap_{j>r}^{s}(Q_i + Q_j) \longrightarrow 0.$$

□

Corollary 1.5. *Let* $I \subset S$ *be a monomial ideal. If bigsize of* I *is one then* I *has minimal depth.*

Proof. We know that $\operatorname{size}(I) \leq \operatorname{bigsize}(I)$. If $\operatorname{size}(I) = 0$ the $\operatorname{depth}(I) = 1$ and the result follows in this case. Now let us suppose that $\operatorname{size}(I) = 1$. By Lemma 1.4 we have $\operatorname{depth}(I) = 2$. Hence the result follows. □

Let $d \geq 2$ be an integer and $\mathcal{M}_d$ the set of monomials of degree d of S. For two monomials $u, v \in \mathcal{M}_d$, with $u \geq_{lex} v$, we consider the lexsegment set

$$\mathcal{L}(u,v) = \{w \in \mathcal{M}_d | u \geq_{lex} w \geq_{lex} v\}.$$

Theorem 1.6. *Let $I = (\mathcal{L}(u,v)) \subset S$ be a lexsegment ideal. Then* $\operatorname{depth}(I) = \operatorname{size}(I) + 1$, *that is I has minimal depth.*

Proof. For the trivial cases $u = v$ the result is obvious. Suppose that $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_1^{b_1} \cdots x_n^{b_n} \in S$. First assume that $b_1 = 0$. If there exist r such that $a_1 = \cdots = a_r = 0$ and $a_{r+1} \neq 0$, then I is a lexsegment ideal in $S' := K[x_{r+1}, \ldots, x_n]$. We get $\operatorname{depth}(IS) = \operatorname{depth}(IS') + r$ and by definition of size we have $\operatorname{size}(IS) = \operatorname{size}(IS') + r$. This means that without loss of generality we can assume that $a_1 > 0$. If $x_n u/x_1 \geq_{lex} v$, then by [5, Proposition 3.2] $\operatorname{depth}(I) = 1$ which implies that $\mathfrak{m} \in \operatorname{Ass}(S/I)$, thus $\operatorname{size}(I) = 0$ and the result follows in this case. Now consider the complementary case $x_n u/x_1 <_{lex} v$, then u is of the form $u = x_1 x_l^{a_l} \cdots x_n^{a_n}$ where $l \geq 2$. Let $I = \cap_{i=1}^{s} Q_i$ be an irredundant primary decomposition of I, where $Q_i's$ are monomial primary ideals. If $l \geq 4$ and $v = x_2^d$ then by [5, Proposition 3.4] we have $\operatorname{depth}(I) = l - 1$. After [10, Proposition 2.5(ii)] we know that

$$\sqrt{\sum_{i=1}^{s} Q_i} = (x_1, x_2, x_l, \ldots, x_n) \notin \operatorname{Ass}(S/I),$$

but $(x_1, x_2), (x_2, x_l, \ldots, x_n) \in \operatorname{Ass}(S/I)$. Therefore, $\operatorname{size}(I) = l - 2$ and we have $\operatorname{depth}(I) = \operatorname{size}(I) + 1$, so we are done in this case. Now consider the case $v = x_2^{d-1} x_j$ for some $3 \leq j \leq n-2$ and $l \geq j + 2$, then again by [5, Proposition 3.4] we have $\operatorname{depth}(I) = l - j + 1$ and by [10, Proposition 2.5(ii)] we have

$$\sqrt{\sum_{i=1}^{s} Q_i} = (x_1, \ldots, x_j, x_l, \ldots, x_n) \notin \operatorname{Ass}(S/I)$$

and $(x_1, \ldots, x_j), (x_2, \ldots, x_j, x_l, \ldots, x_n) \in \operatorname{Ass}(S/I)$. Therefore, $\operatorname{size}(I) = l - j$ and again we have $\operatorname{depth}(I) = \operatorname{size}(I) + 1$. Now for all the remaining cases by [5, Proposition 3.4] we have $\operatorname{depth}(I) = 2$, and by [10, Proposition 2.5(i)]

$$\sqrt{\sum_{i=1}^{s} Q_i} = (x_1, \ldots, x_n) \notin \operatorname{Ass}(S/I),$$

but $(x_1, \ldots, x_j), (x_2, \ldots, x_n) \in \operatorname{Ass}(S/I)$, for some $j \geq 2$. Therefore $\operatorname{size}(I) = 1$. Thus the equality $\operatorname{depth}(I) = \operatorname{size}(I) + 1$ follows in all cases when $b_1 = 0$.

Now let us consider that $b_1 > 0$, then $I = x_1^{b_1} I'$ where $I' = (I : x_1^{b_1})$. Clearly I' is a lexsegment ideal generated by the lexsegment set $\mathcal{L}(u', v')$ where $u' = u/x_1^{b_1}$ and

$v' = v/x_1^{b_1}$. The ideals I, I' are isomorphic, therefore $\mathrm{depth}(I') = \mathrm{depth}(I)$. It is enough to show that $\mathrm{size}(I') = \mathrm{size}(I)$. We have the exact sequence

$$0 \to S/I' \xrightarrow{x_1^{b_1}} S/I \to S/(I, x_1^{b_1}) = S/(x_1^{b_1}) \to 0,$$

and therefore

$$\mathrm{Ass}(S/I') \subset \mathrm{Ass}(S/I) \subset \mathrm{Ass}(S/I') \cup \{(x_1)\}.$$

As $\{(x_1)\} \in \mathrm{Ass}(S/I)$ since it is a minimal prime over I, we get $\mathrm{Ass}(S/I) = \mathrm{Ass}(S/I') \cup \{(x_1)\}$. Let s' be the minimum number such that there exist $P_1, \ldots, P_s \in \mathrm{Ass}\, S/I'$ such that $\sum_{i=1}^{s} P_i = a := \sum_{P \in \mathrm{Ass}(S/I')} P$. Then $\mathrm{size}(I') = s' + \dim(S/a) - 1$. Let s be the minimum number t such that there exist t prime ideals in $\mathrm{Ass}(S/I)$ whose sum is (a, x_1). By [10, Lemma 2.1] we have that atleast one prime ideal from $\mathrm{Ass}(S/I')$ contains necessarily x_1, we have $x_1 \in a$. It follows $s \leq s'$ because anyway $\sum_{i=1}^{s'} P_i = a = \sum_{P \in \mathrm{Ass}(S/I)} P$. If we have $P'_1, \ldots, P'_{s-1} \in \mathrm{Ass}(S/I')$ such that $\sum_{i=1}^{s-1} P'_i + (x_1) = a$ then we have also $\sum_{i=1}^{s-1} P'_i + P_1 = a$ for some $P_1 \in \mathrm{Ass}(S/I')$ which contains x_1. Thus $s = s'$ and so $\mathrm{size}(I) = \mathrm{size}(I')$. □

2. Stanley depth and trivial modifications

Using Corollaries 1.3, 1.5 and Theorems 1.6, 0.2 we get the following theorem.

Theorem 2.1. *Stanley's conjecture holds for I, if it satisfies one of the following statements:*

1. $P_i \not\subseteq \sum_{1=j\neq i}^{s} P_j$ *for all* $i \in [s]$,
2. *the bigsize of I is one,*
3. *I is a lexsegment ideal.*

Remark 2.2. Usually, if Stanley's conjecture holds for an ideal I then we may show that it holds for the module S/I too. There exist no general explanation for this fact. If I is a monomial ideal of bigsize one then Stanley's conjecture holds for S/I. Indeed, case $\mathrm{depth}(S/I) = 0$ is trivial. Suppose $\mathrm{depth}(S/I) \neq 0$, then by Lemma 1.4 $\mathrm{depth}(S/I) = 1$, therefore by [4, Theorem 2.1] $\mathrm{sdepth}(S/I) \geq 1$. If I is a lexsegment ideal then Stanley's conjecture holds for S/I [10]. Below we show this fact in the first case of the above theorem.

Theorem 2.3. *Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant presentation of I as an intersection of primary monomial ideals. Let $P_i := \sqrt{Q_i}$. If $P_i \not\subset \sum_{1=i\neq j}^{s-1} P_j$ for all $i \in [s]$ then* $\mathrm{sdepth}(S/I) \geq \mathrm{depth}(S/I)$, *that is the Stanley's conjecture holds for S/I.*

Proof. Using [6, Lemma 3.6] it is enough to consider the case $\sum_{i=1}^{s} P_i = \mathfrak{m}$. By Proposition 1.2 we have $\mathrm{depth}(S/I) = s - 1$. We show that $\mathrm{sdepth}(S/I) \geq s - 1$. Apply induction on s, case $s = 1$ being clear. Fix $s > 1$ and apply induction on n. If $n \leq 5$ then the result follows by [14]. Let $A := \cup_{i=1}^{s} (G(P_i) \setminus \sum_{1=j\neq i}^{s} G(P_j))$. If $(A) = \mathfrak{m}$ then note that $G(P_i) \cap G(P_j) = \emptyset$ for all $i \neq j$. By [11, Theorem 2.1] we have $\mathrm{sdepth}(S/I) \geq s - 1$. Now suppose that $(A) \neq \mathfrak{m}$. By renumbering the primes and variables we can assume that $x_n \notin A$. There exists a number r,

$2 \leq r \leq s$ such that $x_n \in G(P_j)$, $1 \leq j \leq r$ and $x_n \notin G(P_j)$, $r+1 \leq j \leq s$. Let $S' := K[x_1, \ldots, x_{n-1}]$. First assume that $r < s$. Let $Q'_j = Q_j \cap S'$, $P'_j = P_j \cap S'$ and $J = \bigcap_{i=r+1}^{s} Q'_i \subset S'$, $L = \bigcap_{i=1}^{r} Q'_i \subset S'$. We have $(I, x_n) = ((J \cap L), x_n)$ because $(Q_j, x_n) = (Q'_j, x_n)$ using the structure of monomial primary ideals given in [18]. In the exact sequence

$$0 \longrightarrow S/(I : x_n) \longrightarrow S/I \longrightarrow S/(I, x_n) \longrightarrow 0,$$

the sdepth of the right end is $\geq s-1$ by induction hypothesis on n for $J \cap L \subset S'$ (note that we have $P'_i \not\subset \sum_{1=i\neq j}^{s-1} P'_j$ for all $i \in [s]$ since $x_n \notin A$). Let e_I be the maximum degree in x_n of a monomial from $G(I)$. Apply induction on e_I. If $e_I = 1$ then $(I : x_n) = JS$ and the sdepth of the left end in the above exact sequence is equal with $\operatorname{sdepth}(S/JS) \geq (s-r-1) + r = s-1$ since there are at least r variables which do not divide the minimal monomial generators of ideal $(I : x_n)$ and we may apply induction hypothesis on s for J. By [16, Theorem 3.1] we have $\operatorname{sdepth}(S/I) \geq \min\{\operatorname{sdepth}(S/(I : x_n)), \operatorname{sdepth}(S/(I, x_n))\} \geq s-1$. If $e_I > 1$ then note that $e_{(I:x_n)} < e_I$ and by induction hypothesis on e_I or s we get $\operatorname{sdepth}(S/(I : x_n)) \geq s-1$. As above we obtain by [16, Theorem 3.1] $\operatorname{sdepth}(S/I) \geq s-1$.

Now let $r = s$. If $e_I = 1$ then $I = (L, x_n)$ and by induction on n we have $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(S'/L) \geq s-1$. If $e_I > 1$ then by induction hypothesis on e_I and s we get $\operatorname{sdepth}(S/(I : x_n)) \geq s-1$. As above we are done using [16, Theorem 3.1]. $\square$

Let I be a squarefree monomial ideal with minimal monomial generating set $G(I) = \{u_1, \ldots, u_m\}$. Let u be a monomial of S then $\operatorname{supp}(u) := \{i : x_i \text{ divides } u\}$. Then we call a monomial ideal J a *modification* of I (see [1]), if $G(J) = \{v_1, \ldots, v_m\}$ and $\operatorname{supp}(v_i) = \operatorname{supp}(u_i)$ for all i. Obviously, $\sqrt{J} = I$. Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n$, $a_i \neq 0$ for all i and σ_α be the flat K-morphism of S given by $x_i \to x_i^{a_i}$, $i \in [n]$. Let $I^\alpha := \sigma_\alpha(I)S$. Then I^α is called a *trivial modification* of I. It is well known that $\operatorname{sdepth}(I^\alpha) \leq \operatorname{sdepth}(I)$ by [9, Corollary 2.2]. In our next theorem we will show that the equality holds in this case.

Example 2.4. Let $I = (x_1x_2x_3, x_2x_4, x_4x_5x_6, x_2x_6, x_5x_7, x_1x_2x_6x_7) \subset K[x_1, \ldots, x_7]$ and $\alpha = (2, 3, 6, 3, 7, 8, 2)$, then we have

$$I^\alpha = (x_1^2x_2^3x_3^6, x_2^3x_4^3, x_4^3x_5^7x_6^8, x_2^3x_6^8, x_5^7x_7^2, x_1^2x_2^3x_6^8x_7^2).$$

Lemma 2.5 ([3],[12])**.** *Let r, m and a be positive integers with $r < m$ and $v_1, \ldots, v_m \in K[x_2, \ldots, x_n]$ be some monomials of S. Let $I = (x_1^a v_1, \ldots, x_1^a v_r, v_{r+1}, \ldots, v_m)$ and $I' = (x_1^{a+1} v_1, \ldots, x_1^{a+1} v_r, v_{r+1}, \ldots, v_m)$ be monomial ideals of S. Then*

$$\operatorname{sdepth}(I) = \operatorname{sdepth}(I').$$

Theorem 2.6. *Let $\alpha \in \mathbb{N}^n$, then* $\operatorname{sdepth}(I^\alpha) = \operatorname{sdepth}(I)$.

Proof. Apply induction on $s = \sum_{i=1}^{n} a_i$. If $s = n$ there exist nothing to show. Suppose that $s > n$. Then there exists i such that $a_i > 1$, let us say $a_1 > 1$. Renumbering the variables we may suppose I^α has the form $(x_1^{a_1} v_1, \ldots, x_r^{a_1} v_r, v_{r+1}, \ldots, v_m)$ for some monomials $v_1, \ldots, v_m$ which are not multiple of x_1. By Lemma 2.5 we get $\operatorname{sdepth}(I^\alpha) = \operatorname{sdepth}(J)$ for $J = (x_1^{a_1-1} v_1, \ldots, x_1^{a_1-1} v_r, v_{r+1}, \ldots, x_m)$. But $J = I^{\alpha'}$

for $\alpha' = (a_1 - 1, a_2, \ldots, a_n)$ which has $s' = s - 1$. By induction hypothesis we have $\operatorname{sdepth}(I^{\alpha'}) = \operatorname{sdepth}(I)$, which is enough. $\square$

Corollary 2.7. *Let $I \subset S$ be a squarefree monomial ideal if the Stanley conjecture holds for I, then the Stanley conjecture also holds for I^α.*

Proof. Since $\operatorname{depth}(I) \leq \operatorname{sdepth}(I)$, by [8, Theorem 2.3] and Theorem 2.6 we have $\operatorname{depth}(I^\alpha) \leq \operatorname{depth}(I) \leq \operatorname{sdepth}(I) = \operatorname{sdepth}(I^\alpha)$. This completes the proof. $\square$

References

[1] S. Ahmad, M. Naeem *Cohen-Macaulay monomial ideals with given radical*, Journal of Pure and Applied Algebra (2010), doi:10.1016/j.jpaa.2009.12.024.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay rings*. Rev. ed. Cambridge Studies in Advanced Mathematics 39, Cambridge University Press 1998.

[3] M. Cimpoeaş, *Stanley depth of complete intersection monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie, 51(99) no 3, 2008, 205-211.

[4] M. Cimpoeaş, *Some remarks on the Stanley depth for multigraded modules*, Le Mathematiche, 2008, LXIII, 165-171.

[5] V. Ene, A.Olteanu, L. Sorrenti, *Properties of lexsegment ideals*, Osaka J. Math., **47**, 2010, 1–21.

[6] J. Herzog, M. Vladoiu, X. Zheng, *How to compute the Stanley depth of a monomial ideal*, J. Algebra, 322(9)(2009), 3151-3169.

[7] J. Herzog, D. Popescu, M. Vlădoiu, *Stanley depth and size of a monomial ideal*, to appear in Proceed. AMS.

[8] J. Herzog, Y. Takayama, N. Terai, *On the radical of a monomial ideal*, Arch. Math. (Basel) 85(2005), 397-408.

[9] M. Ishaq, *Upper bounds for the Stanley depth*, Communications in Algebra, **40**(2012), 87-97.

[10] M. Ishaq, *Lexsegment ideals are sequentially Cohen-Macaulay*, to appear in Algebra Colloquium. arXiv:1010.5615v3.

[11] M. Ishaq, *Values and bounds for the Stanley depth*, Carpathian J. Math., **27** (2011), No.2, 217-224.

[12] M. Ishaq, M.I. Qureshi, *Stanley depth and complete k-partite hypergraphs*, arXiv:1104.2412v2. This paper is accepted in Communications in Algebra with changed title *"Upper and lower bounds for the Stanley depth of certain classes of monomial ideals and their residue class rings"*.

[13] G. Lyubeznik, *On the Arithmetical Rank of Monomial ideals*. J. Algebra 112, 8689 (1988).

[14] D. Popescu, *Stanley depth of multigraded modules*, J. Algebra **321**(2009), 2782-2797.

[15] D. Popescu, *Stanley conjecture on intersection of four monomial prime ideals*, arXiv:1009.5646v1.

[16] A. Rauf, *Depth and Stanley depth of multigraded modules*, Comm. Algebra, 38 (2010), 773-784.

[17] R.P. Stanley, *Linear Diophantine equations and local cohomology*, Invent. Math. 68 (1982) 175-193.

[18] R. H. Villarreal, *Monomial Algebras*, New York: Marcel Dekker Inc. 2001.

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